

Probing Planckian physics in de Sitter space with quantum correlations

Jun Feng,¹ Cheng-Yi Sun,² Wen-Li Yang,² Yao-Zhong Zhang,³ and Heng Fan¹

¹*Beijing National Laboratory for Condensed Matter Physics, Institute of Physics,
Chinese Academy of Sciences, Beijing 100190, P. R. China*

²*Institute of Modern Physics, Northwest University, Xian 710069, P. R. China*

³*School of Mathematics and Physics, The University of Queensland, Brisbane, Qld 4072, Australia*

We study the quantum correlations of both free scalar and fermionic field in de Sitter space, while the Planckian modification presented by choice of a particular α -vacuum has been considered. We show the occurrence of degradation of quantum entanglement and quantum discord between field modes for inertial observer in curved space, due to the radiation associated with cosmological horizon. In particular, comparing with standard Bunch-Davies choice, the possible Planckian physics causes some extra decrement on the quantum correlation, which may provides the means to detect quantum gravitational effects via quantum information methodology in future.

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I. INTRODUCTION

Quantum information processing, which is based on principles of quantum mechanics, promises algorithms which surpass their classical counterparts and unconditional secure for quantum communication. Combined with relativity which is another cornerstone of modern physics, a new fast-growing field called Relativistic Quantum Information (RQI) has attracted much attentions in recent years, see Ref.[1] for a review. Besides its importance in such as satellite-based global quantum communication [2] and probing extremely sensitive gravitational effects [3], it is believed that the investigation of quantum information in a relativistic framework can also shed lights on study of black hole information paradox [4] and cosmological evolution [5]. Moreover, it is possible to detect such relativistic quantum effects in laboratory systems like atomic interferometry [6] and circuit QED [7], which makes RQI processing possible for realistic applications.

A novel phenomenon in RQI is that quantum correlations of a mostly entangled physical system are highly observer-dependent. For a bipartite entangled system in flat space, it was shown [8] that an accelerated observer would experience some decrement of the quantum correlations he shared initially with an inertial partner. This is because that the uniformly accelerated observer has no access to information beyond his casual horizon which separates all events in flat space to disconnected Rindler regions. Therefore the information-loss results in so-called Unruh effect [9] which claims the detection of a thermal bath for an accelerated detector in flat space. Such decoherence phenomena should exist for any other kinds of causal horizons related with thermal radiation. For example, in the case of event horizon of black hole, it was shown [10] that a degradation of quantum correlations provoked by Hawking radiation from event horizon would be detected for a static observer nearby the black hole. By a process akin to teleportation but without the classical information transmitted [4, 11], quantum infor-

mation can even escape from a black hole.

As an idealization of inflation regime in cosmology, de Sitter space possesses a cosmological event horizon, from which a thermal spectrum with temperature $T = H/2\pi$ could be detected by a static observer [12]. This so-called Gibbons-Hawking effect results from the highly non-trivial definition of a vacuum state in a time-dependent background where no global timelike Killing vectors could be found. The unique Bunch-Davies vacuum for a comoving observer, matching with Minkowskian vacuum at *arbitrary* short distance, appears as thermal state for a static partner who defines a distinct static vacuum and has no access to the field modes beyond the cosmological horizon [13]. This information loss with Gibbons-Hawking radiation should certainly cause a decoherence phenomenon for an inertial observer in de Sitter space. Since the thermal spectrum described by a same formula as for the temperature of Hawking radiation and Unruh effect, it is wildly believed [14] that, for RQI in de Sitter space, nothing would change in essentially but only the Hubble parameter H replacing the surface gravity of black hole or the acceleration of Unruh detector. However, as we show in this paper, it's *not* the whole story!

The main point is that the definition of Bunch-Davies vacuum, which relies on an ability to follow a field mode to infinitesimal scales, should be broken near some fundamental scale of quantum gravity. Effectively, this means a boundary condition on the vacuum of comoving observer must be imposed when the momentum of field mode \vec{k} cutoff at Planckian scale Λ . Such constraint can also be interpreted as choosing a harmonic oscillator vacuum at the earliest time $\eta_0(\vec{k}) = -\Lambda/H|\vec{k}|$. By the accelerated expansion of de Sitter, those deviation from Bunch-Davies choice would be enormous stretched and lead to a robust signal of H/Λ anisotropy in the cosmic microwave background radiation (CMBR) performed in WMAP and Planck experiments, see Ref. [15] for a review. Therefore, in de Sitter space, any quantum information processing with a particular choice of initial vacuum should include those modifications from Planck scale, since it

has a directly consequence on the behaviors of quantum correlations. Even without a complete knowledge about quantum gravity, such investigation on RQI processing may provide a new typical signature to probe Planckian physics.

Extensive studies show [16] that above boundary conditions can be resolved by selecting a non-trivial de Sitter vacuum state called α -vacuum which respects all the symmetries of spacetime. These vacua are predicted in a free quantum field theory in de Sitter space and can be distinguished by a real number α if the theory consistent with CPT invariance [17]. Nevertheless, among this infinite family, a unique element labeled by $\alpha = -\infty$ could be recognized as the Bunch-Davies state, over which the other vacua $|0^\alpha\rangle$ can be realized as squeezed state. It should be noted [18] that this is only a formal correspondence since each of the vacua $|0^\alpha\rangle$ with different UV behavior is the ground state of a different Hilbert space.

In this paper, we will analyze the quantum correlations of free quantum field in de Sitter space, while the Planckian modification presented by choice of a particular α -vacuum has been considered. For a static observer, these α -vacua become excited and exhibit non-thermal feature derived by the Bogoliubov transformation on his static vacuum. Therefore, in a mostly entangled bipartite system, a deviation from Gibbons-Hawking decoherence should be detected by the inertial observer who is immersed in a bath of non-thermal radiation emanating from the cosmological horizon, while his partner comoving with respect to conformal time see the same entanglement. In this manner, a precise dependence relation between this degradation of quantum correlations and the choice of superselection parameter α would be presented. As the quantum correlations can be used to encode the information of these modifications from Planckian physics, it is important to design a more operational RQI tasks to probe the physics at fundamental scales.

Beside the Planckian modification for bosonic field (e.g. inflaton which drives the inflationary epoch), we also investigate the similar RQI processing for spinor field while fermionic analogous of α -states exist. In inflation regime, such fermions in nonthermal state could couple to inflaton field and has significantly larger loop corrections than that would be expected from the Bunch-Davies choice, even when the inflaton itself is in Bunch-Davies vacuum [19]. For Bunch-Davies choice, similar as in Minkowskian RQI, the quantum correlations between fermionic modes could reach a nonvanishing minima even for infinite spacetime curvature, which means such fermionic state always remains entangled and might be used in real quantum information task in curved spacetime. However, for a non-trivial α -vacuum, we show that extra decrement on the quantum correlations should be included. As Hubble scale increasing infinitely, the remain entanglement would eventually become vanish.

The organization of the paper is as follows. In Sec. II, we review the Planckian physics in de Sitter space by the boundary condition approach; the thermal property

of α -vacua is analyzed and the one-particle excited states for bosonic/fermionic field are obtained. In Sec. III, we investigate the behavior of quantum correlations in static frame with Planckian modifications and show that the pattern of degradation of quantum correlations is depend on the particular choice of initial vacuum. In Sec. IV, we summarise our results and discuss some open problems.

II. PLANCKIAN PHYSICS IN DE SITTER SPACE

A. Boundary condition from Planck scale

While de Sitter space enjoys the same number of isometries as Minkowski space, quantum field theories in this dynamic background is much complicated by the non-trivial definition of unambiguous vacuum. For a quantized free scalar field in de Sitter space with mode expansion

$$\phi(x) = \sum_k [a_k \phi_k(x) + a_{-k}^\dagger \phi_{-k}^*(x)] \quad (1)$$

The vacuum state is defined by $a_k|vac\rangle = 0$ and should respects the spacetime isometries. To specifying mode functions $\phi_k(x)$, one must solve the field equation with some affiliated coordinate systems for different observers.

In inflation regime, a comoving observer adopts planar coordinates which reduce the de Sitter metric as

$$ds^2 = \frac{1}{(H\eta)^2} (-d\eta^2 + d\rho^2 + \rho^2 d\Omega^2) \quad (2)$$

where $\eta = -e^{-Ht}/H$ is conformal time. The coordinates cover the upper right triangle of the Carter-Penrose diagram (both region I and II), as depicted in Fig.1.

Since the space is always accelerated expanding, it follows that the wavelength of field mode could become arbitrary smaller than the curvature radius if one goes backwards in conformal time far enough. In the limit $\eta \rightarrow -\infty$, the mode functions reduce to the adiabatic modes defined in flat space and satisfy $\partial_\eta \phi(\eta, \vec{x}) = -ik\phi(\eta, \vec{x})$, which means any distinction between de Sitter space and Minkowski space should be suppressed then. The corresponding Bunch-Davies vacuum $a_k(\eta)|0, \eta\rangle = 0$, which matches the conformal vacuum of Minkowski space in same limit, therefore becomes essentially unique. Since its invariance under de Sitter isometries [20], the Bunch-Davies vacuum is always chose as initial state to estimate the primordial power spectrum of inflationary perturbations.

However, as we mentioned before, the existence of fundamental Planck scale, where the unknown quantum gravitational effects become important, prevents us from following a mode back unlimited. To estimate the latest time with Planckian physics [15], a cutoff on physical momentum could be made as $p = ka(\eta) = \Lambda$, where Λ refers to Planckian energy scale. With scale factor

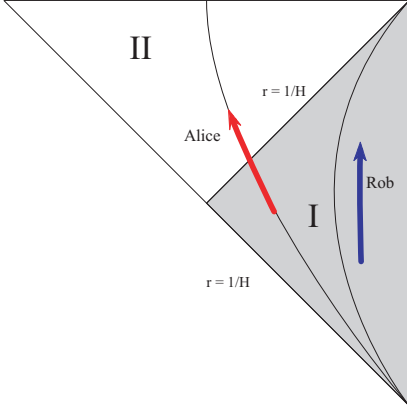


FIG. 1: The Penrose-Carter diagram of de Sitter space. The static observer situated at $r = 0$ would observe the event horizon at $r = H^{-1}$. We consider a bipartite system of Alice and Rob comoving with respect to conformal time initially. If Rob turns to be static in region I while Alice maintains its move with respect to η , a degradation of quantum correlation due to Gibbons-Hawking effect should be observed for Rob.

$a(\eta) = -1/H\eta$ read from (2), one has the conformal time as

$$\eta_0 = -\frac{\Lambda}{Hk} \quad (3)$$

when the boundary condition is imposed to redefine the vacuum state for the comoving observer as $a_k(\eta_0)|0, \eta_0\rangle = 0$.

It should be noted that this new vacuum is in general different from Bunch-Davies (but recover the later by taking $\eta_0 \rightarrow -\infty$) and is a direct result of new physics at Planck scale. Moreover, since the mode expansion (1) admits both vacua, this new vacuum could be formally realized as a squeezed state over Bunch-Davies like

$$|0, \eta_0\rangle = S|0^\infty\rangle \quad (4)$$

where the superscript ∞ always indicates Bunch-Davies choice in this paper. Without a complete theory of quantum gravity, this boundary condition on vacuum of comoving observer can still provide a typical signature of Planck physics by modify the spectrum of quantum fluctuations after inflation. For instance, it was shown [16] that the power spectrum $P(k) \sim \langle |\phi_k|^2 \rangle$ with respect to the new vacuum would be modified like $\Delta P(k)/P(k) = \frac{H}{\Lambda} \sin \frac{2\Lambda}{H}$, which is expected to be observed in WMAP or Planck satellite experiments.

More ambitious view is that above argument indeed provides an one-parameter family of vacua with the single parameter given by the fundamental scale (Planckian or stringy). Equivalently, this allows us to discuss the boundary condition from quantum gravity in terms of so-called α -vacua which have been known for a long time [17, 19].

Consider a new set of mode basis related with Bunch-Davies one by Mottola-Allen (MA) transformation

$$\phi_k^\alpha(\eta, \vec{x}) = N_\alpha[\phi_k^\infty(\eta, \vec{x}) + e^\alpha \phi_{-k}^{\infty*}(\eta, \vec{x})] \quad (5)$$

where α is an arbitrary complex number with $\text{Re}\alpha < 0$, $N_\alpha = 1/\sqrt{1 - e^{\alpha+\alpha^*}}$. The one-parameter family of vacuum are defined as $a_k^\alpha|0^\alpha\rangle = 0$, where

$$a_k^\alpha = N_\alpha[a_k^\infty - e^{\alpha^*} a_{-k}^{\infty\dagger}] \quad (6)$$

are corresponding annihilation operators. These α -vacua preserve all $SO(1,4)$ de Sitter isometries. As $\text{Re}\alpha \rightarrow -\infty$, $a_k^\alpha \rightarrow a_k^\infty$ which indicates the Bunch-Davies vacuum is included in this one-parameter family of vacua. Moreover, the boundary condition (4) can now be resolved as

$$|0_k^\alpha\rangle = \exp[\alpha(a_k^{\infty\dagger} a_{-k}^{\infty\dagger} - a_{-k}^\infty a_k^\infty)]|0_k^\infty\rangle \quad (7)$$

Since the time-reversal invariance of theory requires α be real, we henceforth adopt $\alpha = \text{Re}\alpha$ for simplicity.

Similarly, a fermionic analogous of α -vacua could be defined for a free massive fermionic field with mode expansion

$$\psi(x) = \sum_k [b_k \psi_k^+(\eta, \vec{x}) + c_k^\dagger \psi_k^-(\eta, \vec{x})] \quad (8)$$

where the creation and annihilation operators obey the anticommutation relations as $\{b_k, b_{k'}^\dagger\} = \{c_k, c_{k'}^\dagger\} = \delta_{k,k'}$. The Bunch-Davies choice for fermionic modes read as $b_k^\infty|0_k^\infty\rangle = c_k^\infty|0_k^\infty\rangle = 0$. With the fermionic MA transformations imposed for both particle and anti-particle operators

$$\begin{aligned} b_k^\alpha &= \tilde{N}_\alpha[b_k^\infty - e^{\alpha^*} c_{-k}^{\infty\dagger}] \\ c_k^\alpha &= \tilde{N}_\alpha[c_k^\infty + e^{\alpha^*} b_{-k}^{\infty\dagger}] \end{aligned} \quad (9)$$

where $\tilde{N}_\alpha = 1/\sqrt{1 + e^{\alpha+\alpha^*}}$. The fermionic α -vacua are defined by $b_k^\alpha|0_k^\alpha\rangle = c_k^\alpha|0_k^\alpha\rangle = 0$, which are also invariant under de Sitter isometries. The boundary condition now becomes

$$|0_k^\alpha\rangle = \exp[\alpha(b_k^{\infty\dagger} c_{-k}^{\infty\dagger} + b_{-k}^\infty c_k^\infty)]|0_k^\infty\rangle \quad (10)$$

Strictly speaking, only a small set of de Sitter α -vacua could be chosen as an alternative initial state rather than Bunch-Davies. The value of α is constrained by the fundamental scale putting in possible quantum gravity model. From (3), (7) and (10), it could be estimated [15]

$$e^\alpha \sim \frac{H}{\Lambda} \quad (11)$$

B. Thermality and one-particle state from α -vacua

While the Bunch-Davies vacuum appears thermal with $T = H/2\pi$ for static observer, it is clear that both bosonic and fermionic α -vacua would exhibit some non-thermal feature since the corrections from MA transformations (6) and (9). Such deviations from pure thermal spectrum are believed to provide a robust signal from Planckian physics in power spectrum of fluctuation after inflation [16]. To investigate how these modifications affect RQI processing, we need to obtain the excited states

of bosonic/fermionic modes during the particle creation near the cosmological horizon. By standard Bogoliubov Transformation Technology (BTT), the Bogoliubov coefficients, relating the mode functions in comoving frame and static frame, are usually highly involved even in low dimensions [21]. Fortunately, for our purposes here we only need to employ basic Bose/Fermi statistics and Gibbons-Hawking relation.

For an inertial observer with static coordinates, de Sitter metric becomes

$$ds^2 = -(1 - r^2 H^2) dt^2 + (1 - r^2 H^2)^{-1} dr^2 + r^2 d\Omega^2 \quad (12)$$

which covers half of the region of conformal coordinates denoted as region I in Fig. 1. The hypersurface on $r = 1/H$ is a cosmological casual horizon for an observer situated at $r = 0$. Since the existence of the Killing vector ∂_t , the quantum field modes can be properly separated into positive and negative frequency parts for static observer along the trajectory of Killing vector with respect to cosmological time t . However, it should be noted [8, 22] that to construct a complete basis of field expansion in static coordinates one needs to combine another set of modes defined in region II. For scalar field, we denote a_k^I and a_k^{II} as the particle annihilation operators in region I and II. For fermionic field, b_k^I and b_k^{II} denote particle annihilation operators in region I and II, while c_k^I and c_k^{II} corresponding anti-particle operators. Since the field modes in two coordinates (2) and (12) do not coincident, the associated annihilator operators can be related with each other by Bogoliubov transformations.

For a scalar field, it reads

$$a_k^\infty = \cosh r a_k^I - \sinh r a_{-k}^{II\dagger} \quad (13)$$

With squeezing operator $S(r) = \exp[r(a_k^{I\dagger} a_{-k}^{II\dagger} - a_k^{II} a_{-k}^I)]$, the Bunch-Davies vacuum for comoving observer in conformal time η can be realized as squeezed states

$$|0_k^\infty\rangle = \text{sech } r \sum_{n=0}^{\infty} \tanh^n r |n_k^I; n_{-k}^{II}\rangle \quad (14)$$

where $\tanh^2 r = \exp(-2\pi|k|/H)$ known from Gibbons-Hawking effect. This means that particles created in pairs one on either side of event horizon, only one in Region I should be detected as de Sitter radiation by inertial observer.

For general α -vacua, additional deviations from thermality should be included. From (7) and (13), it follows that

$$|0_k^\alpha\rangle = \sqrt{1 - \tanh^2 r \Delta^2} \sum_{n=0}^{\infty} \tanh^n r \Delta^n |n_k^I; n_{-k}^{II}\rangle \quad (15)$$

where

$$\Delta \equiv \frac{1 + e^\alpha \tanh^{-1} r}{1 + e^\alpha \tanh r} = \frac{1 + e^{\alpha + \pi|k|/H}}{1 + e^{\alpha - \pi|k|/H}} \quad (16)$$

As $\alpha \rightarrow -\infty$, these corrections can be neglected and a pure thermal de Sitter radiation is recovered. The one-particle excitation in α -vacua can also be obtained as

$$|1_k^\alpha\rangle = \left[1 - \tanh^2 r \Delta^2\right] \sum_{n=0}^{\infty} \tanh^n r \Delta^n \sqrt{n+1} |n_k^I; n_{-k}^{II}\rangle \quad (17)$$

For a fermionic field, the Bogoliubov transformations give

$$\begin{aligned} b_k^\infty &= \cos \tilde{r} b_k^I - \sin \tilde{r} c_{-k}^{II\dagger} \\ c_k^{\infty\dagger} &= \cos \tilde{r} c_k^{II\dagger} + \sin \tilde{r} b_{-k}^I \end{aligned} \quad (18)$$

Because of the Pauli exclusion principle, there are only two allowed states for each mode, $|0\rangle^+$ and $|1\rangle^+$ for particles, and similarly for antiparticles. Then the fermionic Bunch-Davies vacuum for particles becomes an excited state in static frame as

$$\begin{aligned} |0_k^\infty\rangle^+ &= \cos \tilde{r} \sum_{n=0}^1 \tan^n \tilde{r} |n_k^I; n_{-k}^{II}\rangle^+ \\ &= \cos \tilde{r} |0_k^I, 0_{-k}^{II}\rangle^+ + \sin \tilde{r} |1_k^I, 1_{-k}^{II}\rangle^+ \end{aligned} \quad (19)$$

where $\tan^2 \tilde{r} = \exp(-2\pi|k|/H)$ known from Gibbons-Hawking effect. From (10) and (18), for a general α -state of particle, we have

$$|0_k^\alpha\rangle^+ = \frac{1}{\sqrt{1 + \tan \tilde{r} \tilde{\Delta}}} \sum_{n=0}^1 \tan^n \tilde{r} \tilde{\Delta}^n |n_k^I; n_{-k}^{II}\rangle^+ \quad (20)$$

where

$$\tilde{\Delta} \equiv \frac{1 + e^\alpha \tan^{-1} \tilde{r}}{1 - e^\alpha \tan \tilde{r}} = \frac{1 + e^{\alpha + \pi|k|/H}}{1 - e^{\alpha - \pi|k|/H}} \quad (21)$$

The disparity between above correction from (16) is a result of different statistics possessed by bosonic and fermionic field. The one-particle excitation in fermionic α -vacua is

$$|1_k^\alpha\rangle^+ = |1_k^\infty\rangle^+ = |1_k^I\rangle |0_{-k}^{II}\rangle \quad (22)$$

In above analysis, we have illustrated that the α -vacua, which resolve the boundary condition (4) from physics at Planck scale, do exhibit the nonthermal feature for both bosonic and fermionic field as shown in (15) and (20). Since the field modes in region II beyond cosmological event horizon are inaccessible for static observer in region I, the related degree of freedom should be traced over. The main point is such information-loss should also suffer these nonthermal corrections from possible Planckian physics, which make the behavior the quantum correlations among field modes be very different from our expectation.

III. QUANTUM CORRELATIONS OF FREE FIELDS

A. quantum correlations of scalar field

The simplest way to illustrate the influence of α -vacua on quantum information task is to consider a maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0_A^{(\alpha)}\rangle|0_R^{(\alpha)}\rangle + |1_A^{(\alpha)}\rangle|1_R^{(\alpha)}\rangle) \quad (23)$$

shared by two comoving observers Alice and Rob in coordinates (2) with respect to conformal time η . As Rob turning to be static, the corresponding states should be changed into those in static coordinates. From (15), we have

$$\begin{aligned} \rho_{A,RI} = & \sum_{n=0}^{\infty} \frac{1}{2} \tanh^{2n} r \Delta^{2n} (1 - \tanh^2 r \Delta^2) \left[|0n\rangle\langle 0n| \right. \\ & + \sqrt{(n+1)(1 - \tanh^2 r \Delta^2)} (|0n\rangle\langle 1n+1| + |1n+1\rangle\langle 0n|) \\ & \left. + (n+1)(1 - \tanh^2 r \Delta^2) \times |1n+1\rangle\langle 1n+1| \right] \quad (24) \end{aligned}$$

To estimate the quantum correlations, we should calculate the negativity [23] which is an entanglement monotone defined as the sum of negative eigenvalues of partial transposed density matrix for (24). It follows that

$$\begin{aligned} N_{A,RI} = & \sum_{n=0}^{+\infty} \frac{1}{4} \tanh^{2n} r \Delta^{2n} (1 - \tanh^2 r \Delta^2) \\ & \left| \sqrt{[\tanh^2 r \Delta^2 + n(\frac{\coth^2 r}{\Delta^2} - 1)]^2 + 4(1 - \tanh^2 r \Delta^2)} \right. \\ & \left. - \tanh^2 r \Delta^2 + n(\frac{1 - \coth^2 r}{\Delta^2}) \right| \quad (25) \end{aligned}$$

which is a function of α and Hubble scale.

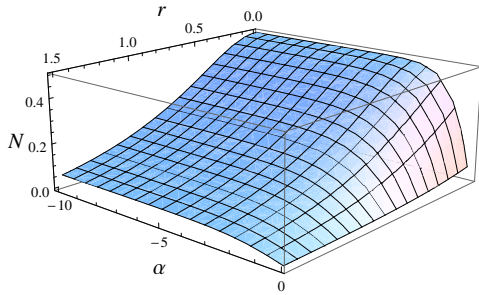


FIG. 2: The negativity of Alice-RobI system as the function of Hubble scale and α .

Our first observation is that a degradation of quantum entanglement occurs for static Rob (see Fig.2) as we expected. This phenomenon roots from the information-loss via de Sitter radiation detected by inertial observer

similar as Unruh effect for accelerated frame in flat space. For Bunch-Davies state which means $\alpha = -\infty$, the spectrum is pure thermal. If $H \rightarrow 0$ ($r \rightarrow 0$), de Sitter space approaches flat with infinite large curvature radius $l \equiv 1/H \rightarrow \infty$, and the negativity (25) would approach 0.5 due to vanish radiation then. On the other hand, in the limit of infinite curvature $H \rightarrow \infty$ ($r \rightarrow \infty$)¹, the state has no distillable entanglement since negativity is exactly zero.

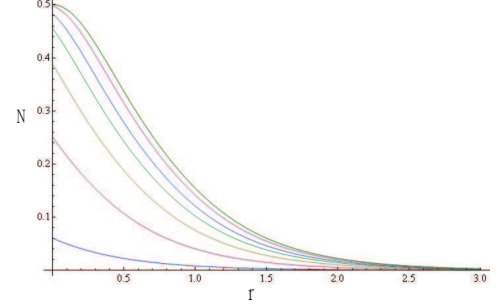


FIG. 3: The negativity encodes the modifications from fundamental scale. The curves from bottom to top correspond to different choice of vacua with $\alpha = -0.1, -0.5, -1, -2, -5, -100$. For $\alpha \rightarrow -\infty$, negativity decay starts from 0.5, and is completely suppressed when $\alpha \rightarrow 0^-$.

The unexpected feature of entanglement in de Sitter space, which is the key result of this paper, is that the Planckian modification represented by vacua ambiguity could be directly encoded in quantum correlations measured by static observer Rob. Comparing with the standard Bunch-Davies choice, the negativity is suppressed for all α -vacua choice with $\alpha \neq -\infty$. In a realistic model [16] with $H \sim 10^{14}\text{GeV}$ and $\Lambda \sim 10^{16}\text{GeV}$, we can estimate the typical value of α from (11) as $\alpha \sim -4.6$. This choice of initial vacuum results a significantly modification in the degradation of entanglement for Rob as depicted in Fig.3. Moreover, since the quantum entanglement encodes the possible Planckian physics, we expect that, in principle, one can probe the unknown physics at Planck scale by some quantum information tasks.

It should be remarked that, unlike in Bunch-Davies choice, for a non-trivial α -vacuum, even in the limit $r \rightarrow 0$ where de Sitter space approaching flat, the negativity is still smaller than 0.5 which means the distillable entanglement suffers a decrement for static observer. At first glance, this seems contradict with RQI in flat space [8] where an inertial observer should maintain his amounts of entanglement. However, since α -vacua become squeezed states over Bunch-Davies vacuum which matches the conformal vacuum of flat space in

¹ Strictly speaking, the Hubble scale could not grow unlimited if a cutoff at Planck length is also made on the curvature radius of space $l = L_P$. However, since $H = \frac{1}{L_P} \gg 1$ then, we can safely ignore this constrain in our analysis.

Minkowskian limit, state (23) with $\alpha \neq -\infty$ now appears mixed for inertial observer Rob. This analysis shows that our result is indeed consistent with those arguments of RQI in flat space.

We now estimate the mutual information which measures the total classical and quantum correlations in Alice-Rob system, $I_{A,RI} = S(\rho_A) + S(\rho_{RI}) - S(\rho_{A,RI})$, where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. A straightforward calculation gives

$$I_{A,RI} = 1 - \frac{1}{2} \log_2 (\tanh^2 r \Delta^2) - \frac{1}{2} (1 - \tanh^2 r \Delta^2) \sum_{n=0}^{\infty} \tanh^{2n} r \Delta^{2n} \left\{ \left(1 - n + n \frac{\coth^2 r}{\Delta^2} \right) \log_2 \left(1 - n + n \frac{\coth^2 r}{\Delta^2} \right) - [n + 2 - (n + 1) \tanh^2 r \Delta^2] \times \log_2 [n + 2 - (n + 1) \tanh^2 r \Delta^2] \right\} \quad (26)$$

which is a function of α and Hubble scale. As the space curvature approaching infinite $H \rightarrow \infty$, $I \rightarrow 1$, most quantum correlations have decayed. The mutual information should also encode different vacua selected by Alice-Rob initially. Therefore, we have a set of mutual information evolution trajectories labeled by α as illustrated in Fig.4.

Since (23) is pure, one reads that $S(\rho_{A,RI}) = S(\rho_{RII})$ and $S(\rho_{A,RII}) = S(\rho_{RI})$. Therefore the combined mutual information, $I_{A,RI} + I_{A,RII} = 2$, is conserved which suggests a correlation transfer between Alice-RobI and Alice-RobII systems and be independent with initial vacua selection.

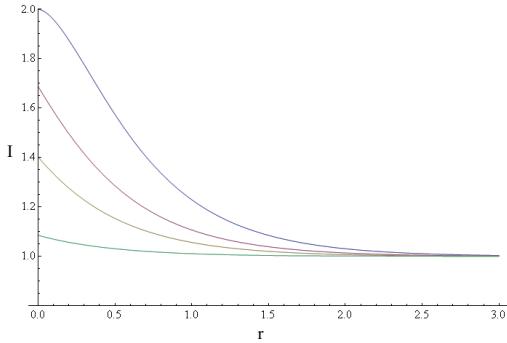


FIG. 4: Mutual information of Alice-RobI for fixed $\alpha = -0.1, -0.5, -1, -10$ (from bottom to top).

As learned from RQI in Minkowski space [8], there could be the residual quantum correlation measured by quantum discord of system exist even at infinite acceleration limit. Similar results should also be expected in de Sitter space when the distillable entanglement vanish. However, as we stated before, the behavior of quantum correlation is also influenced by initial vacuum ambiguity from possible Planckian physics. Therefore, it becomes necessary to specify the quantum discord evolution for distinct selected initial vacua.

To determine the quantum discord [24–26] which measures pure quantum correlations left beside classical part in mutual information $D(A : RI) = I(A : RI) - C(A : RI)$, we first measure the subsystem of Alice of $\rho_{A,RI}$ by a complete set of projectors $\{\Pi_{\pm} = \frac{I_1 \pm \vec{x} \cdot \vec{\sigma}}{2}\}$, where \vec{x} is parameterized as $(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The post-measurement density matrix becomes $\rho_{RI|\pm} = \text{Tr}(\Pi_j \rho_{A,RI} \Pi_j) / p_j$, and $p_j = \text{Tr}_{A,RI}(\Pi_j \rho_{A,RI} \Pi_j) = 1/2$ for our case

$$\rho_{RI|\pm} = \sum_{n=0}^{\infty} \frac{1}{2} \tanh^{2n} r \Delta^{2n} (1 - \tanh^2 r \Delta^2) \left[(1 \pm \cos \theta) \times |n\rangle\langle n| \pm \sin \theta \sqrt{(n+1)(1 - \tanh^2 r \Delta^2)} \times (|n\rangle\langle n+1| + |n+1\rangle\langle n|) + (1 \mp \cos \theta)(n+1) \times (1 - \tanh^2 r \Delta^2) |n+1\rangle\langle n+1| \right] \quad (27)$$

have nonvanish eigenvalues

$$\lambda_{\pm} = \left\{ \frac{1}{2} \tanh^{2n} r \Delta^{2n} (1 - \tanh^2 r \Delta^2) [1 \pm \cos \theta + (1 \mp \cos \theta)(n+1)(1 - \tanh^2 r \Delta^2)] \right\}_{n=0}^{\infty} \quad (28)$$

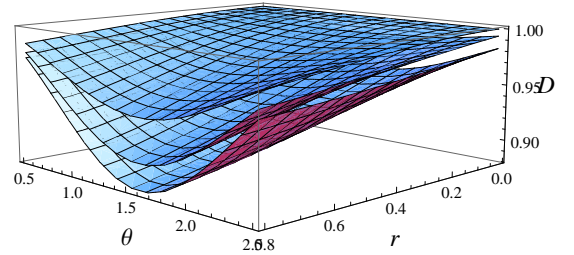


FIG. 5: Quantum discords of Alice-RobI as functions of θ and Hubble scale. Three hypersurfaces from top to bottom correspond to $\alpha = -20, -1, -0.5$ respectively.

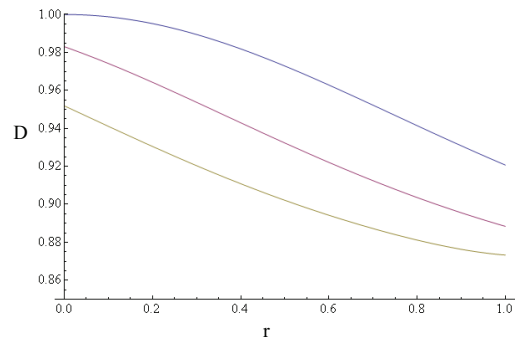


FIG. 6: Quantum discords of Alice-RobI as functions of Hubble scale. Three curves from top to bottom correspond to $\alpha = -20, -1, -0.5$ respectively.

By minimize the conditional entropy $S_{\{\Pi_j\}}(RI|A) = \sum_j p_j S(\rho_{RI|\pm})$, one can rewrite the quantum discord as $D = S(\rho_A) - S(\rho_{ARI}) + \min_{\{\Pi_j\}} S_{\{\Pi_j\}}(RI|A)$ and yield

$$D(\theta, r, \alpha) = 1 + \frac{1}{2}(1 - \tanh^2 r \Delta^2) \sum_{n=0}^{\infty} \left\{ \tanh^{2n} r \Delta^{2n} \times [1 + (n+1)(1 - \tanh^2 r \Delta^2)] \log_2 \left[\frac{1}{2}(1 - \tanh^2 r \Delta^2) \times \tanh^{2n} r \Delta^{2n} [1 + (n+1)(1 - \tanh^2 r \Delta^2)] \right] \right\} - \max_{i=\pm} \frac{1}{2} \sum \text{Tr}(\lambda_i \log_2 \lambda_i) \quad (29)$$

where λ_i is in the eigenvalues spectrum of $\rho_{RI|\pm}$. As plotted in Fig.5 for three distinct initial vacua with $\alpha = -20, -1, -0.5$, the common minima appears at $\theta = \frac{\pi}{2}$. Therefore the quantum discord for Alice-RobI system is $D(\theta = \frac{\pi}{2}, r, \alpha)$, depicted in Fig.6.

We read that the quantum discord decay as H growing but reaches a nonvanishing minimal value when $H \rightarrow \infty$, similar as the case for accelerated observer in Minkowski space. For $\alpha \neq -\infty$, where the possible Planckian physics impose a different initial vacuum, the static observer should detect a bigger decrement of quantum discord than that of Bunch-Davies choice. Since quantum discord can also be used as physical resource of certain quantum information task [27], it may provide another operational method to probe Planckian physics in future.

B. quantum correlations of fermionic field

Since all matter fields in nature are fermionic, it is important to extend above results to quantum correlations between fermionic field modes. For this purpose, we assume two comoving observers Alice and Rob with respect to conformal time η share a maximally entangled initial state as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|0_A^{(\alpha)}\rangle^+ |0_R^{(\alpha)}\rangle^+ + |1_A^{(\alpha)}\rangle^+ |1_R^{(\alpha)}\rangle^+] \quad (30)$$

While Rob turning to be static, the field modes beyond cosmological horizon become causal disconnected for him. Therefore, the initial pure state appear as a mixed state for Rob by tracing over the degrees of region II. It follows that

$$\begin{aligned} \rho_{A,RI} = & \frac{1}{2}[(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)^{-1} |00\rangle\langle 00| \\ & + (\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)^{-\frac{1}{2}} (|00\rangle\langle 11| + |11\rangle\langle 00|) \\ & + (\frac{\cot^2 \tilde{r}}{\tilde{\Delta}^2} + 1)^{-1} |01\rangle\langle 01| + |11\rangle\langle 11|] \end{aligned} \quad (31)$$

We have seen that such fermionic system is much simpler than the bosonic case due to its statistics. Similar as

before, we use negativity to quantify entanglement, which gives

$$N_{A,RI} = \frac{1}{2}(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)^{-1} \quad (32)$$

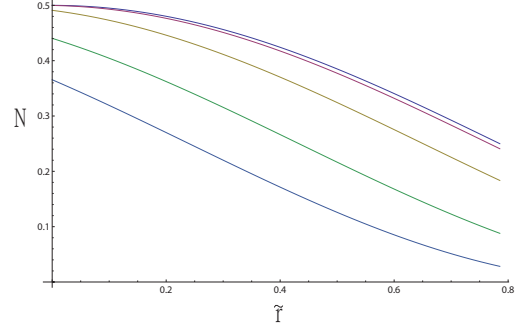


FIG. 7: Negativity with modifications from Planckian physics. The curves from bottom to top correspond to different choice of vacua with $\alpha = -0.5, -1, -2, -5, -10$.

For $\alpha = -\infty$, Rob recovers the Bunch-Davies choice. In Minkowskian limit $H = 0$ ($\tilde{r} = 0$) where space approaching flat, the state (30) remains maximally entangled. As the space curvature grows infinitely $H \rightarrow \infty$ ($\tilde{r} \rightarrow \frac{\pi}{4}$), using Fermi statistics $\tan \tilde{r} = \exp(-\pi|k|/H)$, the negativity is $\frac{1}{4}$ which means the state (30) will always preserve some degree of distillable entanglement that can be used in quantum information task.

For $\alpha \neq -\infty$, the quantum entanglement is much suppressed compared with Bunch-Davies case. As depicted in Fig.8, the negativity is smaller than 0.5 even for Minkowskian limit since the α -vacua then become squeezed states over Minkowskian vacuum. Moreover, when $H \rightarrow \infty$, less degree of distillable entanglement can be preserved for Rob than in Bunch-Davies case. For $\alpha \rightarrow 0^-$, the residual entanglement at limit $H \rightarrow \infty$ becomes vanish which is very different with the fermionic RQI in flat space.

The total correlations between Alice and Rob is measured by mutual information, which quantifies the information two correlated observers possess about the state of each other. Using (31), the von Neumann entropy of Alice-RobI system is

$$\begin{aligned} S(\rho_{A,RI}) = & -\frac{\tan^2 \tilde{r} \tilde{\Delta}^2 + 2}{2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)} \log_2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 2) \\ & - \frac{\tan^2 \tilde{r} \tilde{\Delta}^2}{2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)} \log_2(\tan^2 \tilde{r} \tilde{\Delta}^2) \\ & + \log_2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1) + 1 \end{aligned} \quad (33)$$

Then the mutual information of the state (31) is given

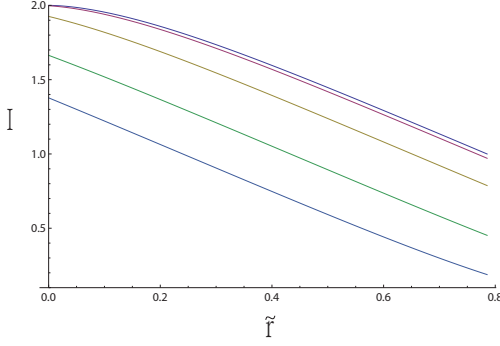


FIG. 8: Mutual information with modifications from Planckian physics. The curves from bottom to top correspond to different choice of vacua with $\alpha = -0.5, -1, -2, -5, -10$.

by

$$\begin{aligned}
 I_{A,RI} = & 1 + \frac{\tan^2 \tilde{r} \tilde{\Delta}^2}{2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)} \log_2 \tan^2 \tilde{r} \tilde{\Delta}^2 \\
 & + \frac{\tan^2 \tilde{r} \tilde{\Delta}^2 + 2}{2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)} \log_2 (\tan^2 \tilde{r} \tilde{\Delta}^2 + 2) \\
 & - \frac{2 \tan^2 \tilde{r} \tilde{\Delta}^2 + 1}{2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)} \log_2 (2 \tan^2 \tilde{r} \tilde{\Delta}^2 + 1) \quad (34)
 \end{aligned}$$

Similar as bosonic case, since $S(\rho_{A,RI}) = S(\rho_{RII})$ and $S(\rho_{A,RII}) = S(\rho_{RI})$, therefore the combined mutual information, $I_{A,RI} + I_{A,RII} = 2$, is conserved quantity which suggests a correlation transfer between Alice-RobI and Alice-RobII systems should be independent with initial vacua selection.

To derive the pure quantum correlation between Alice and Rob, we make the measurements on Alice by projector Π_{\pm} defined before. After the measurement, the quantum state (31) becomes

$$\begin{aligned}
 \rho_{RI|\pm} = & \frac{1}{2} \{ (1 \pm \cos \theta) (\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)^{-1} |0\rangle\langle 0| \\
 & \pm \sin \theta (\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)^{-\frac{1}{2}} (|0\rangle\langle 1| + |1\rangle\langle 0|) \\
 & + [1 \mp \cos \theta + (1 \pm (\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)^{-\frac{1}{2}})] |1\rangle\langle 1| \} \quad (35)
 \end{aligned}$$

where $p_{\pm} = \text{Tr} \Pi_{\pm} \rho_{A,RI} \Pi_{\pm} = 1/2$. The eigenvalues of these density matrices are

$$\begin{aligned}
 \lambda_+ = & \frac{1}{2} (1 \pm \sqrt{1 \pm \frac{4 \tan^2 \tilde{r} \tilde{\Delta}^2 \cos^4 \frac{\theta}{2}}{(1 + \tan^2 \tilde{r} \tilde{\Delta}^2)^2}}) \\
 \lambda_- = & \frac{1}{2} (1 \pm \sqrt{1 \pm \frac{4 \tan^2 \tilde{r} \tilde{\Delta}^2 \sin^4 \frac{\theta}{2}}{(1 + \tan^2 \tilde{r} \tilde{\Delta}^2)^2}}) \quad (36)
 \end{aligned}$$

The value of the quantum discord is related to minima

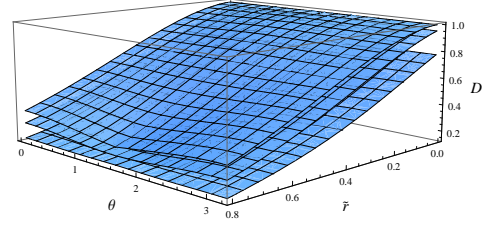


FIG. 9: Fermionic quantum discords of Alice-RobI as functions of θ and Hubble scale. Three hypersurfaces from top to bottom correspond to $\alpha = -10, -2, -1$ respectively.

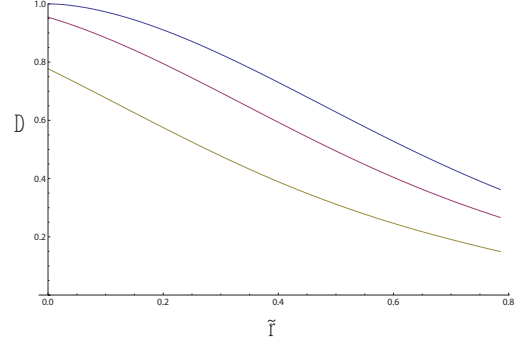


FIG. 10: Fermionic quantum discords of Alice-RobI as functions of Hubble scale. Three curves from top to bottom correspond to $\alpha = -10, -2, -1$ respectively.

of conditional entropy

$$\begin{aligned}
 D(\theta, \tilde{r}, \alpha) = & \frac{\tan^2 \tilde{r} \tilde{\Delta}^2 + 2}{2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)} \log_2 (\tan^2 \tilde{r} \tilde{\Delta}^2 + 2) \\
 & + \frac{\tan^2 \tilde{r} \tilde{\Delta}^2}{2(\tan^2 \tilde{r} \tilde{\Delta}^2 + 1)} \log_2 \tan^2 \tilde{r} \tilde{\Delta}^2 - \log_2 (\tan^2 \tilde{r} \tilde{\Delta}^2 + 1) \\
 & - \max_{iI\pm} \frac{1}{2} \sum \text{Tr}(\lambda_i \log \lambda_i) \quad (37)
 \end{aligned}$$

where λ_i is in the eigenvalues spectrum of $\rho_{RI|\pm}$. As plotted in Fig.9 for three distinct initial vacua with $\alpha = -10, -2, -1$, the common minima appears at $\theta = \frac{\pi}{2}$. Therefore the quantum discord for Alice-RobI system is $D(\theta = \frac{\pi}{2}, r, \alpha)$, depicted in Fig.10. Similar as bosonic case, we read different patterns of change on quantum discord between fermionic modes label by distinct initial α -vacua imposed by unknown physics at Planck scale. As spacetime curvature growing, the pure quantum correlation would decrease since the information loss from (non-)thermal Gibbons-Hawking effect.

IV. CONCLUSION AND DISCUSSION

It has been known for a long time that quantum correlations, in particular quantum entanglement, not only plays a key role in quantum information science, but can also provide powerful means for other research areas such

as condensed matter physics (see [28] for example). In this paper, we extend this universality to the physics at some fundamental scales, like Planckian or stringy, by its influence on quantum correlation in certain quantum information tasks. We analyze the quantum correlations of free fields in de Sitter space, while the Planckian modifications presented by vacuum ambiguity has been considered. We show that the corrections from fundamental scale are encoded in patterns of degradation of quantum correlations, like quantum entanglement and quantum discord, for a static observer in de Sitter space. Comparing with standard Bunch-Davies choice, the possible Planckian physics cause some extra decrement on the quantum correlation, which may provides the means to detect quantum gravitational effects via quantum information methodology.

The design of such intelligent quantum information experiments is of course the next challenging open problem. Nevertheless, one can at least simulate these Planckian modifications by analogue gravity experiments, like using ion trap. In detector picture [29], replacing the conformal time η with the experimenter's clock time, the detector should evolve with respect to the simulated proper time t which is equal to the cosmic time. To simulate vacuum ambiguity, the detector's response function represents the probability for an ion excited by interaction with the field. For general $\alpha \neq -\infty$, the Wightman function is $G_\alpha^+(\xi, \xi') = N_\alpha [G_E^+(\xi, \xi') + e^{\alpha+\alpha^*} G_E^+(\xi', \xi) + e^\alpha G_E^+(-\xi, \xi') + e^{\alpha^*} G_E^+(\xi', -\xi)]$. Therefore the ion ana-

logue of this function $\langle \phi_m(\xi) \phi_m(\xi') \rangle$ should be evaluated in some motional-state[30], since the α -vacua can be interpreted as squeezed states over Bunch-Davies vacuum state.

Non-locality of Planckian physics in de Sitter space is another important open problem. From above analysis, we observe that there could be nonvanish quantum correlations even between the field modes, which are separated by cosmological horizon, by tracing over all degrees of Alice in (23) and (30). For RQI in flat space, such kind of quantum correlations still lacks physical meaning in terms of information theory [31]. However, the quantum field theory in de Sitter space do exhibits some nonlocal feature from Planckian physics, since the Green function in a non-trivial α -vacuum contains singular correlations between antipodal points (see for example Ref.[32]). Therefore, detailed analysis on such problem in quantum information approach may confirm the conjecture that non-locality should be an essential feature for a unitary quantum gravity theory.

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